Final exam 2010/2011 second semester

Q1. Rewrite in if-then form:-

a) sami well be allowed on racing board only if he is an expert

p

p only if $q \equiv p \to q$

(*) if sami is allowed on racing board then he is an expert

b) A sufficient condition for Jameela's team to win the championship is that her team wins the remaining games

s

r

r is sufficient condition for s

$r \to s$

(*) if Jameelas team win the remaining games, then Jameelas team wins the championship.

c) being divisible by 3 is a necessery condition for this number to be divisible by 9.

r s $s \to r$

(*) if this number is disible by 9, then it is divisible by 3.

if

d) Fareed will go to market whenever it dose not rain.

p q

(*) if it dose not rain, then Fareed will go to market

Q3: ① $\sim p \to r \wedge \sim s$ $\therefore r \wedge \sim s$ $\therefore r$ $\therefore \sim s$

② $t \to s$ $\therefore \sim t$ ✓

③ $u \to \sim p$ $\therefore \sim p$

④ $\sim w$

⑤ $u \vee w$ $\therefore u$

$\therefore \sim t$

Q4: Define $f: \mathbb{Z} \to \mathbb{Z}^+$ on $f(x) = |x| + 1$

a) what is the domain of f:

$\mathbb{Z}$

- what is the codomain of f:

$\mathbb{Z}^+$

- what is the range of f:

$\mathbb{Z}^+$

b) Is f onto?

on-to $\longleftrightarrow$ $\forall y \in \mathbb{Z}^+ \, \exists x \in \mathbb{Z} \quad y = |x| + 1$

$|x| = -1 + y \in \mathbb{Z} \quad \forall y \in \mathbb{Z}^+$

- one-to-one?

$\forall x_1, x_2 \in \mathbb{Z} \quad f(x_1) = f(x_2) \longrightarrow x_1 = x_2$

Let $f(x_1) = f(x_2)$

$|x_1| + 1 = |x_2| + 1$

$|x_1| = |x_2|$

for $x_1 = -1 \in \mathbb{Z}$, $x_2 = 1 \in \mathbb{Z}$

$|x_1| = |x_2|$ but $x_1 \neq x_2$

Q5: A sequence $A_0, A_1, \ldots$ is defined as

$$A_0 = 4$$
$$A_n = 4A_{n-1} + 3n \cdot 2^n, \quad n \geq 1$$

show that:

$$A_n = 10 \cdot 4^n - (3n+6) \cdot 2^n, \quad n \geq 0$$

Given:

$$A_0 = 4$$
$$A_n = 4A_{n-1} + 3n \cdot 2^n, \quad n \geq 1$$

Let $p(n)$: $A_n = 10 \cdot 4^n - (3n+6) \cdot 2^n, \quad n \geq 0$

basis: to show $p(0)$

For $n=0 \Rightarrow A_n = 10 \cdot 4^n - (3n+6) 2^n = A_0 = 10 \cdot 4^0 - (6) \cdot 2^0$
$= 10 - 6 = 4$

$\therefore p(0)$.

Assumption: Let:

$p(k)$: $A_k = 10 \cdot 4^k - (3k+6) \cdot 2^k, \quad k \geq 0$

Induction: To show:

$p(k+1)$: $A_{k+1} = 10 \cdot 4^{k+1} - (3(k+1)+6) \cdot 2^{k+1}, \quad k \geq 0$

From given:

from assumption.

$$A_{k+1} = 4A_k + 3(k+1) \cdot 2^{k+1}$$
$$= 4\left[10 \cdot 4^k - (3k+6) \cdot 2^k\right] + 3(k+1) \cdot 2^{k+1}, \quad k \geq 0$$
$$= 10 \cdot 4^{k+1} - (3k+6) \cdot 2^{k+2} + 3(k+1) \cdot 2^{k+1}$$
$$= 10 \cdot 4^{k+1} + \left[2^{k+1}\left(-2(3k+6) + 3k+3\right)\right]$$
$$= 10 \cdot 4^{k+1} + \left[2^{k+1}(-6k-12+3k+3)\right]$$
$$= 10 \cdot 4^{k+1} + \left[2^{k+1}(-3k-3-6)\right]$$
$$= 10 \cdot 4^{k+1} + \left[2^{k+1} - (3(k+1)+6)\right]$$
$$= 10 \cdot 4^{k+1} - (3(k+1)+6) \cdot 2^{k+1}$$

$\therefore p(k+1)$

Hence proved.

Q6: prove that if X is real and $X^2 - X - 2 > 0$ then $X < -1$ or $X > 2$

$$\forall x \in R \quad x^2 - x - 2 > 0 \longrightarrow (X < -1) \vee (X > 2)$$

Contrapositive:

$$\forall x \in R \quad \sim[(X < -1) \vee (X > 2)] \longrightarrow \sim[x^2 - x - 2 > 0]$$

$$\forall x \in R \quad (X \geq -1) \wedge (X \leq 2) \longrightarrow X^2 - X - 2 \leq 0$$

Let $(X \geq -1) \wedge (X \leq 2)$

$\therefore$ $(X + 1) \geq 0$ ——— (1)

$(X - 2) \leq 0$ ——— (2)

multiplay :-

$$(X + 1)(X - 2) \leq 0$$

$$x^2 - x - 2 \leq 0$$

$\therefore$ proved.

Q7: Let $A = \mathbb{Z}$ and R on A is defined as:

$$\forall a,b \in A \quad aRb \longleftrightarrow 5a - 2b = 3n \quad , n \in \mathbb{Z}$$

a) Show that R is an equivelance relation.

b) List 4 elements of $[1]$.

Reflexive: To show:

$$aRa \longleftrightarrow 5a - 2a = 3n$$

$$\longleftrightarrow 3a = 3n$$

$$\therefore n = a \in \mathbb{Z} \quad T$$

$\therefore aRa$

Hence reflexive

Symmetric: To show:

$$\forall a,b \in \mathbb{Z} \quad aRb \longrightarrow bRa$$

To assume: aRb

To conclude: $bRa \longleftrightarrow 5b - 2a = 3n_1 \quad , n_1 \in \mathbb{Z}$

Let $a,b \in \mathbb{Z} \wedge aRb$

$$\therefore 5a - 2b = 3n \quad , n \in \mathbb{Z} \quad \text{— (1)}$$

$$(5a - 2b) + (5b - 2a) = 5a - 2b + 5b - 2a = 3a + 3b$$

(From assumption) $3n + (5b - 2a) = 3a + 3b$ from (1)

$$\therefore 5b - 2a = 3a + 3b - 3n$$

$$= 3(a + b - n)$$

$$= 3n \quad , \quad n = a + b - n \in \mathbb{Z}$$

$\therefore bRa$

Hence symmetric

transitive: To show:

$\forall a,b,c \in Z \quad aRb \wedge bRC \longrightarrow aAC$

to assumes $aRb \wedge bRC$

to conclude: $aRC \longleftrightarrow 5a-2c=3n, \; n \in Z$

Let $aRb \wedge bRC$

$\therefore \; aRb$ — ①

bRC — ②

$\therefore \; 5a-2b=3n_1, \; n_1 \in Z$ — ①

$5b-2c=3n_2, \; n_2 \in Z$ — ②

adding:-

$5a-2b+5b-2c=3n_1+3n_2$

$(5a-2c)+3b=3n_1+3n_2$

$\therefore \; 5a-2c=3n_1+3n_2-3b$

$=3(n_1+n_2-b)$

$=3n, \quad n=n_1+n_2-b \in Z$

$\therefore$ transitive.

$\therefore$ equivelance relation.

$[1]=\{x \in Z \mid xR1\}$

$=\{x \in Z \mid 5x-2.1=3n, \; n \in Z\}$

$=\{x \in Z \mid 5x-2=3n, \; n \in Z\}$

$=\{1, 4, 7, 10, \ldots\}$